4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Team Round

1. [**2**] Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

Answer: $\boxed{0}$ We claim that our integer is divisible by 3 for all positive integers x. Indeed, we have

$$\begin{aligned} 3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x &\equiv (0)^x + (-1)^x + (1)^x + (-1)^x + (1)^x + (-1)^x + (1)^x \\ &\equiv 3[(1)^x + (-1)^x] \\ &\equiv 0 \pmod 3. \end{aligned}$$

It is clear that for all $x \geq 1$, our integer is strictly greater than 3, so it will always be composite, making our answer 0.

2. [**4**] Determine the set of all real numbers p for which the polynomial $Q(x) = x^3 + px^2 - px - 1$ has three distinct real roots.

Answer: $\boxed{p > 1 \text{ and } p < -3}$ First, we note that

$$x^3 + px^2 - px - 1 = (x-1)(x^2 + (p+1)x + 1).$$

Hence, $x^2 + (p+1)x + 1$ has two distinct roots. Consequently, the discriminant of this equation must be positive, so $(p+1)^2 - 4 > 0$, so either $p > 1$ or $p < -3$. However, the problem specifies that the quadratic must have *distinct* roots (since the original cubic has distinct roots), so to finish, we need to check that 1 is not a double root–we will do this by checking that 1 is not a root of $x^2 + (p+1)x + 1$ for any value p in our range. But this is clear, since $1 + (p+1) + 1 = 0 \Rightarrow p = -3$, which is not in the aforementioned range. Thus, our answer is all p satisfying $p > 1$ or $p < -3$.

3. [**6**] Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$.

Answer: $\boxed{-3193}$ Define $Q(x) = P(x) - x - 6 = x^4 - 29x^3 + ax^2 + (b-1)x + (c-6)$ and notice that $Q(5) = Q(11) = Q(17) = 0$. $Q(x)$ has degree 4 and by Vieta's Formulas the sum of its roots is 29, so its last root is $29 - 17 - 11 - 5 = -4$, giving us $Q(x) = (x-5)(x-11)(x-17)(x+4)$. This means that $P(1) = Q(1) + 7 = (-4)(-10)(-16)(5) + 7 = -3200 + 7 = -3193$.

4. [**7**] Determine the number of quadratic polynomials $P(x) = p_1x^2 + p_2x - p_3$, where p_1, p_2, p_3 are not necessarily distinct (positive) prime numbers less than 50, whose roots are distinct rational numbers.

Answer: $\boxed{31}$ The existence of distinct rational roots means that the given quadratic splits into linear factors. Then, since p_1, p_3 are both prime, we get that the following are the only possible factorizations:

- $(p_1x - p_3)(x+1) \Rightarrow p_2 = p_1 - p_3$
- $(p_1x + p_3)(x-1) \Rightarrow p_2 = -p_1 + p_3$
- $(p_1x - 1)(x + p_3) \Rightarrow p_2 = p_1p_3 - 1$
- $(p_1x + 1)(x - p_3) \Rightarrow p_2 = -p_1p_3 + 1$

In the first case, observe that since $p_2 + p_3 = p_1$, we have $p_1 > 2$, so p_1 is odd and exactly one of p_2, p_3 is equal to 2. Thus, we get a solutions for every pair of twin primes below 50, which we enumerate to be $(3,5), (5,7), (11,13), (17,19), (29,31), (41,43)$, giving 12 solutions in total. Similarly, the second case gives $p_1 + p_2 = p_3$, for another 12 solutions.

In the third case, if p_1, p_3 are both odd, then p_2 is even and thus equal to 2. However, this gives $p_1 p_3 = 3$, which is impossible. Therefore, at least one of p_1, p_3 is equal to 2. If $p_1 = 2$, we get $p_2 = 2p_3 - 1$, which we find has 4 solutions: $(p_2, p_3) = (3, 2), (5, 3), (13, 7), (37, 19)$. Similarly, there are four solutions with $p_3 = 2$. However, we count the solution $(p_1, p_2, p_3) = (2, 3, 2)$ twice, so we have a total of 7 solutions in this case. Finally, in the last case

$$p_2 = -p_1 p_3 + 1 < -(2)(2) + 1 < 0,$$

so there are no solutions. Hence, we have a total of $12 + 12 + 7 = 31$ solutions.

5. [**3**] Sixteen wooden Cs are placed in a 4-by-4 grid, all with the same orientation, and each is to be colored either red or blue. A *quadrant operation* on the grid consists of choosing one of the four two-by-two subgrids of Cs found at the corners of the grid and moving each C in the subgrid to the adjacent square in the subgrid that is 90 degrees away in the clockwise direction, without changing the orientation of the C. Given that two colorings are the considered same if and only if one can be obtained from the other by a series of quadrant operations, determine the number of distinct colorings of the Cs.

C	C	C	C
C	C	C	C
C	C	C	C
C	C	C	C

Answer: $\boxed{1296}$ For each quadrant, we have three distinct cases based on the number of Cs in each color:

- *Case 1:* all four the same color: 2 configurations (all red or all blue)
- *Case 2:* 3 of one color, 1 of the other: 2 configurations (three red or three blue)
- *Case 3:* 2 of each color: 2 configurations (red squares adjacent or opposite)

Thus, since there are 4 quadrants, there are a total of $(2 + 2 + 2)^4 = 1296$ possible grids.

6. [**5**] Ten Cs are written in a row. Some Cs are upper-case and some are lower-case, and each is written in one of two colors, green and yellow. It is given that there is at least one lower-case C, at least one green C, and at least one C that is both upper-case and yellow. Furthermore, no lower-case C can be followed by an upper-case C, and no yellow C can be followed by a green C. In how many ways can the Cs be written?

Answer: $\boxed{36}$ By the conditions of the problem, we must pick some point in the line where the green Cs transition to yellow, and some point where the upper-case Cs transition to lower-case. We see that the first transition must occur before the second, and that they cannot occur on the same C. Hence, the answer is $\binom{9}{2} = 36$.

7. [**7**] Julia is learning how to write the letter C. She has 6 differently-colored crayons, and wants to write Cc Cc Cc Cc Cc. In how many ways can she write the ten Cs, in such a way that each upper case C is a different color, each lower case C is a different color, and in each pair the upper case C and lower case C are different colors?

Answer: $\boxed{222480}$ Suppose Julia writes Cc a sixth time, coloring the upper-case C with the unique color different from that of the first five upper-case Cs, and doing the same with the lower-case C (note: we allow the sixth upper-case C and lower-case c to be the same color). Note that because the colors on the last Cc are forced, and any forced coloring of them is admissible, our problem is equivalent to coloring these six pairs.

There are 6! ways for Julia to color the upper-case Cs. We have two cases for coloring the lower-case Cs:

- *Case 1:* the last pair of Cs use two different colors. In this case, all six lower-case Cs have a different color to their associated upper-case C, and in addition the six lower-case Cs all use each color exactly once. In other words, we have a derangement* of the six colors, based on the colors of the upper-case Cs. We calculate $D_6 = 265$ ways to color the lower-case Cs here.
- *Case 2:* the last pair of Cs have both Cs the same color. Then, the color of the last lower-case C is forced, and with the other five Cs we, in a similar way to before, have a derangement of the remaining five colors based on the colors of the first five lower-case Cs, so we have $D_5 = 44$ ways to finish the coloring.

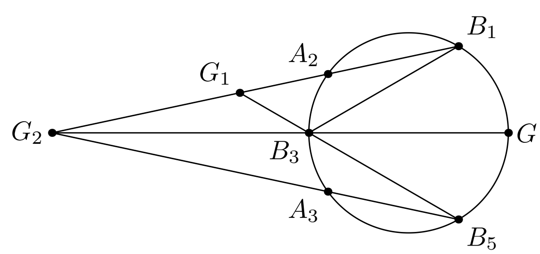
Our answer is thus $720(265 + 44) = 222480$.

*A **derangement** is a permutation π of the set $\{1, 2, \ldots, n\}$ such that $\pi(k) \neq k$ for all k, i.e. there are no fixed points of the permutation. To calculate D_n, the number of derangements of an n-element set, we can use an inclusion-exclusion argument. There are $n!$ ways to permute the elements of the set. Now, we subtract the number of permutations with at least one fixed point, which is $\binom{n}{1}(n-1)! = \frac{n!}{1!}$, since we choose a fixed point, then permute the other $n-1$ elements. Correcting for overcounting, we add back the number of permutations with at least two fixed points, which is $\binom{n}{2}(n-2)! = \frac{n!}{2!}$. Continuing in this fashion by use of the principle of inclusion-exclusion, we get

$$D_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right).$$

8. [**4**] Let $G, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, B_5$ be ten points on a circle such that $GA_1A_2A_3A_4$ is a regular pentagon and $GB_1B_2B_3B_4B_5$ is a regular hexagon, and B_1 lies on minor arc GA_1. Let B_5B_3 intersect B_1A_2 at G_1, and let B_5A_3 intersect GB_3 at G_2. Determine the degree measure of $\angle GG_2G_1$.

   Answer: $\boxed{12^\circ}$

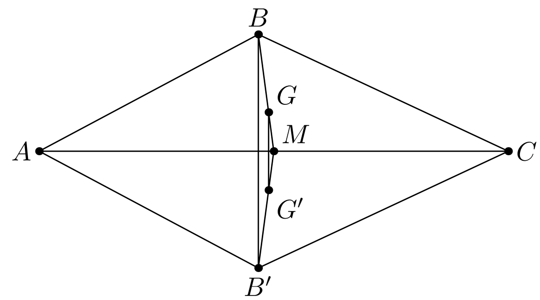


Note that GB_3 is a diameter of the circle. As a result, A_2, A_3 are symmetric with respect to GB_3, as are B_1, B_5. Therefore, B_1A_2 and B_5A_3 intersect along line GB_3, so in fact, B_1, A_2, G_1, G_2 are collinear. We now have

$$\angle GG_2G_1 = \angle GG_2B_1 = \frac{\widehat{GB_1} - \widehat{B_3A_2}}{2} = \frac{60^\circ - 36^\circ}{2} = 12^\circ.$$

9. [**4**] Let ABC be a triangle with $AB = 9$, $BC = 10$, and $CA = 17$. Let B' be the reflection of the point B over the line CA. Let G be the centroid of triangle ABC, and let G' be the centroid of triangle $AB'C$. Determine the length of segment GG'.

Answer: $\boxed{\frac{48}{17}}$



Let M be the midpoint of AC. For any triangle, we know that the centroid is located $2/3$ of the way from the vertex, so we have $MG/MB = MG'/MB' = 1/3$, and it follows that $MGG' \sim MBB'$. Thus, $GG' = BB'/3$. However, note that BB' is twice the altitude to AC in triangle ABC. To finish, we calculate the area of ABC in two different ways. By Heron's Formula, we have

$$[ABC] = \sqrt{18(18-9)(18-10)(18-17)} = 36,$$

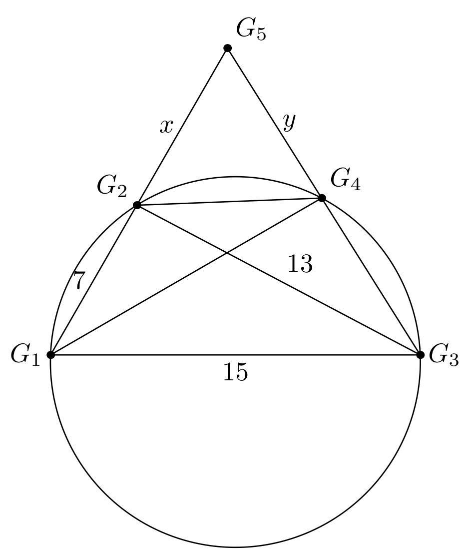
and we also have

$$[ABC] = \tfrac{1}{4}BB' \cdot AC = \tfrac{17}{4}(BB'),$$

from which it follows that $GG' = BB'/3 = 48/17$.

10. [**8**] Let $G_1G_2G_3$ be a triangle with $G_1G_2 = 7$, $G_2G_3 = 13$, and $G_3G_1 = 15$. Let G_4 be a point outside triangle $G_1G_2G_3$ so that ray $\overrightarrow{G_1G_4}$ cuts through the interior of the triangle, $G_3G_4 = G_4G_2$, and $\angle G_3G_1G_4 = 30°$. Let G_3G_4 and G_1G_2 meet at G_5. Determine the length of segment G_2G_5.

Answer: $\boxed{\frac{169}{23}}$



We first show that quadrilateral $G_1G_2G_4G_3$ is cyclic. Note that by the law of cosines,

$$\cos \angle G_2G_1G_3 = \frac{7^2 + 15^2 - 13^2}{2 \cdot 7 \cdot 15} = \frac{1}{2},$$

so $\angle G_2G_1G_3 = 60°$. However, we know that $\angle G_3G_1G_4 = 30°$, so G_1G_4 is an angle bisector. Now, let G_1G_4 intersect the circumcircle of triangle $G_1G_2G_3$ at X. Then, the minor arcs $\widehat{G_2X}$ and $\widehat{G_3X}$ are subtended by the equal angles $\angle G_2G_1X$ and $\angle G_3G_1X$, implying that $G_2X = G_3X$, i.e. X is on the perpendicular bisector of G_2G_3, l. Similarly, since $G_4G_2 = G_4G_3$, G_4 lies on l. However, since l and G_1G_4 are distinct (in particular, G_1 lies on G_1G_4 but not l), we in fact have $X = G_4$, so $G_1G_2G_4G_3$ is cyclic.

We now have $G_5G_2G_4 \sim G_5G_3G_1$ since $G_1G_2G_4G_3$ is cyclic. Now, we have $\angle G_4G_3G_2 = \angle G_4G_1G_2 = 30°$, and we may now compute $G_2G_4 = G_4G_3 = 13/\sqrt{3}$. Let $G_5G_2 = x$ and $G_5G_4 = y$. Now, from $G_5G_4G_2 \sim G_5G_1G_3$, we have:

$$\frac{x}{y + 13/\sqrt{3}} = \frac{13/\sqrt{3}}{15} = \frac{y}{x + 7}.$$

Equating the first and second expressions and cross-multiplying, we get

$$y + \frac{13\sqrt{3}}{3} = \frac{15\sqrt{3}x}{13}.$$

Now, equating the first and third expressions and and substituting gives

$$\left(\frac{15\sqrt{3}x}{13} - \frac{13\sqrt{3}}{3}\right)\left(\frac{15\sqrt{3}x}{13}\right) = x(x + 7).$$

Upon dividing both sides by x, we obtain a linear equation from which we can solve to get $x = 169/23$.