4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Theme Round

1. [**3**] Five of James' friends are sitting around a circular table to play a game of Fish. James chooses a place between two of his friends to pull up a chair and sit. Then, the six friends divide themselves into two disjoint teams, with each team consisting of three consecutive players at the table. If the order in which the three members of a team sit does not matter, how many possible (unordered) pairs of teams are possible?

   Answer: $\boxed{5}$ Note that the team not containing James must consist of three consecutive players who are already seated. We have 5 choices for the player sitting furthest clockwise on the team of which James is not a part. The choice of this player uniquely determines the teams, so we have a total of 5 possible pairs.

2. [**3**] In a game of Fish, R2 and R3 are each holding a positive number of cards so that they are collectively holding a total of 24 cards. Each player gives an integer estimate for the number of cards he is holding, such that each estimate is an integer between 80% of his actual number of cards and 120% of his actual number of cards, inclusive. Find the smallest possible sum of the two estimates.

   Answer: $\boxed{20}$ To minimize the sum, we want each player to say an estimate as small as possible–i.e. an estimate as close to 80% of his actual number of cards as possible. We claim that the minimum possible sum is 20.

   First, this is achievable when R2 has 10 cards and estimates 8, and when R3 has 14 cards and estimates 12.

   Then, suppose that R2 has x cards and R3 has $24 - x$. Then, the sum of their estimates is

   $$\left\lceil \frac{4}{5}(x) \right\rceil + \left\lceil \frac{4}{5}(24-x) \right\rceil \geq \left\lceil \frac{4}{5}(x) + \frac{4}{5}(24-x) \right\rceil \geq \left\lceil \frac{4}{5}(24) \right\rceil \geq 20.$$

   Note: We use the fact that for all real numbers a, b, $\lceil a \rceil + \lceil b \rceil \geq \lceil a + b \rceil$.

3. [**5**] In preparation for a game of Fish, Carl must deal 48 cards to 6 players. For each card that he deals, he runs through the entirety of the following process:

   1. He gives a card to a random player.
   2. A player Z is randomly chosen from the set of players who have at least as many cards as every other player (i.e. Z has the most cards or is tied for having the most cards).
   3. A player D is randomly chosen from the set of players other than Z who have at most as many cards as every other player (i.e. D has the fewest cards or is tied for having the fewest cards).
   4. Z gives one card to D.

   He repeats steps 1-4 for each card dealt, including the last card. After all the cards have been dealt, what is the probability that each player has exactly 8 cards?

   Answer: $\boxed{\frac{5}{6}}$ After any number of cards are dealt, we see that the difference between the number of cards that any two players hold is at most one. Thus, after the first 47 cards have been dealt, there is only one possible distribution: there must be 5 players with 8 cards and 1 player with 7 cards. We have two cases:

   - Carl gives the last card to the player with 7 cards. Then, this player must give a card to another, leading to a uneven distribution of cards.
   - Carl gives the last card to a player already with 8 cards. Then, that player must give a card to another; however, our criteria specify that he can only give it to the player with 7 cards, leading to an even distribution.

The probability of the second case happening, as Carl deals at random, is $\frac{5}{6}$.

4. [**6**] Toward the end of a game of Fish, the 2 through 7 of spades, inclusive, remain in the hands of three distinguishable players: DBR, RB, and DB, such that each player has at least one card. If it is known that DBR either has more than one card or has an even-numbered spade, or both, in how many ways can the players' hands be distributed?

   Answer: $\boxed{450}$ First, we count the number of distributions where each player has at least 1 card. The possible distributions are:

   - *Case 1:* 4/1/1: There are 3 choices for who gets 4 cards, 6 choices for the card that one of the single-card players holds, and 5 choices for the card the other single-card player holds, or $3 \times 6 \times 5 = 90$ choices.
   - *Case 2:* 3/2/1: There are 6 choices for the single card, $\binom{5}{2} = 10$ choices for the pair of cards, and $3! = 6$ choices for which player gets how many cards, for a total of $6 \times 10 \times 6 = 360$ choices.
   - *Case 3:* 2/2/2: There are $\binom{6}{2} = 15$ choices for the cards DBR gets, $\binom{4}{2} = 6$ for the cards that RB gets, and DB gets the remaining two cards. This gives a total of $15 \times 6 = 90$ choices.

   Thus, we have a total of $90 + 360 + 90 = 540$ ways for the cards to be distributed so that each person holds at least one.

   Next, we look at the number of ways that the condition cannot be satisfied if each player has at least one card. Then, DBR must have no more than one card, and cannot have an even spade. We only care about cases where he has a non-zero number of cards, so he must have exactly 1 odd spade. Then, we see that there are $2^5 - 2 = 30$ ways to distribute the other 5 cards among RB and DB so that neither has 0 cards. Since there are 3 odd spades, this gives 3×30 bad cases, so we have $540 - 90 = 450$ ones where all the problem conditions hold.

5. [**8**] For any finite sequence of positive integers π, let $S(\pi)$ be the number of strictly increasing sub-sequences in π with length 2 or more. For example, in the sequence $\pi = \{3, 1, 2, 4\}$, there are five increasing sub-sequences: $\{3, 4\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, and $\{1, 2, 4\}$, so $S(\pi) = 5$. In an eight-player game of Fish, Joy is dealt six cards of distinct values, which she puts in a random order π from left to right in her hand. Determine

$$\sum_{\pi} S(\pi),$$

   where the sum is taken over all possible orders π of the card values.

   Answer: $\boxed{8287}$ For each subset of Joy's set of cards, we compute the number of orders of cards in which the cards in the subset are arranged in increasing order. When we sum over all subsets of Joy's cards, we will obtain the desired sum.

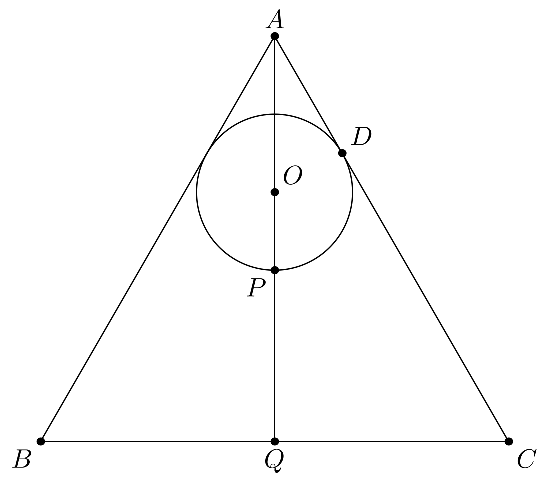
   Consider any subset of k cards. The probability that they are arranged in increasing order is precisely $1/k!$ (since we can form a $k!$-to-1 correspondence between all possible orders and orders in which the cards in our subset are in increasing order), and there are $6! = 720$ total arrangements so exactly $720/k!$ of them give an increasing subsequence in the specified cards. Now for any for $k = 2, 3, 4, 5, 6$, we have $\binom{6}{k}$ subsets of k cards, so we sum to get

$$\sum_{\pi} S(\pi) = \sum_{k=2}^{6} \binom{6}{k} \cdot \frac{6!}{k!} = 8287.$$

6. [**3**] Let ABC be an equilateral triangle with $AB = 3$. Circle ω with diameter 1 is drawn inside the triangle such that it is tangent to sides AB and AC. Let P be a point on ω and Q be a point on segment BC. Find the minimum possible length of the segment PQ.

Answer: $\boxed{\frac{3\sqrt{3}-3}{2}}$ Let P, Q, be the points which minimize the distance. We see that we want both to lie on the altitude from A to BC. Hence, Q is the foot of the altitude from A to BC and $AQ = \frac{3\sqrt{3}}{2}$. Let O, which must also lie on this line, be the center of ω, and let D be the point of tangency between ω and AC. Then, since $OD = \frac{1}{2}$, we have $AO = 2OD = 1$ because $\angle OAD = 30°$, and $OP = \frac{1}{2}$. Consequently,

$$PQ = AQ - AO - OP = \frac{3\sqrt{3}-3}{2}.$$



7. [**4**] Let XYZ be a triangle with $\angle XYZ = 40°$ and $\angle YZX = 60°$. A circle Γ, centered at the point I, lies inside triangle XYZ and is tangent to all three sides of the triangle. Let A be the point of tangency of Γ with YZ, and let ray $\overrightarrow{XI}$ intersect side YZ at B. Determine the measure of $\angle AIB$.

Answer: $\boxed{10°}$ Let D be the foot of the perpendicular from X to YZ. Since I is the incenter and A the point of tangency, $IA \perp YZ$, so

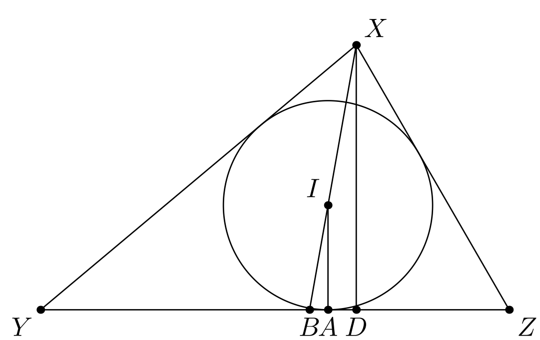
$$AI \parallel XD \Rightarrow \angle AIB = \angle DXB.$$

Since I is the incenter,

$$\angle BXZ = \frac{1}{2}\angle YXZ = \frac{1}{2}(180° - 40° - 60°) = 40°.$$

Consequently, we get that

$$\angle AIB = \angle DXB = \angle ZXB - \angle ZXD = 40° - (90° - 60°) = 10°.$$

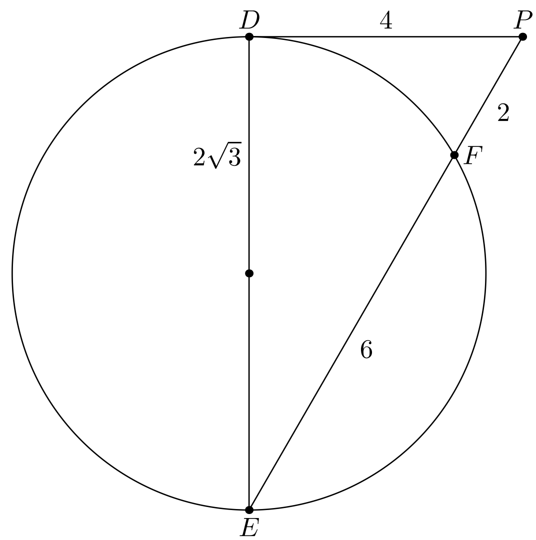


8. [**5**] Points D, E, F lie on circle O such that the line tangent to O at D intersects ray $\overrightarrow{EF}$ at P. Given that $PD = 4$, $PF = 2$, and $\angle FPD = 60°$, determine the area of circle O.

Answer: $\boxed{12\pi}$ By the power of a point on P, we get that

$$16 = PD^2 = (PF)(PE) = 2(PE) \Rightarrow PE = 8.$$

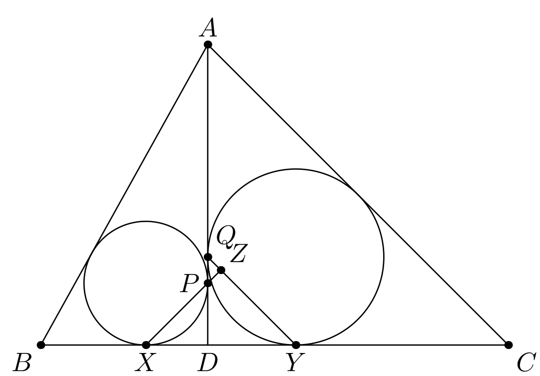
However, since $PE = 2PD$ and $\angle FPD = 60°$, we notice that PDE is a $30-60-90$ triangle, so $DE = 4\sqrt{3}$ and we have $ED \perp DP$. It follows that DE is a diameter of the circle, since tangents the tangent at D must be perpendicular to the radius containing D. Hence, the area of the circle is $(\frac{1}{2}DE)^2\pi = 12\pi$.



9. [**6**] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. The inscribed circles of triangles ABD and ACD are tangent to AD at P and Q, respectively, and are tangent to BC at X and Y, respectively. Let PX and QY meet at Z. Determine the area of triangle XYZ.

   Answer: $\boxed{\frac{25}{4}}$ First, note that $AD = 12, BD = 5, CD = 9$.

   By equal tangents, we get that $PD = DX$, so PDX is isosceles. Because D is a right angle, we get that $\angle PXD = 45°$. Similarly, $\angle XYZ = 45°$, so XYZ is an isosceles right triangle with hypotenuse XY. However, by tangents to the incircle, we get that $XD = \frac{1}{2}(12+5-13) = 2$ and $YD = \frac{1}{2}(12+9-15) = 3$. Hence, the area of the XYZ is $\frac{1}{4}(XY)^2 = \frac{1}{4}(2+3)^2 = \frac{25}{4}$.



10. [**7**] Let Ω be a circle of radius 8 centered at point O, and let M be a point on Ω. Let S be the set of points P such that P is contained within Ω, or such that there exists some rectangle $ABCD$ containing P whose center is on Ω with $AB = 4$, $BC = 5$, and $BC \parallel OM$. Find the area of S.

Answer: $\boxed{164 + 64\pi}$ We wish to consider the union of all rectangles $ABCD$ with $AB = 4$, $BC = 5$, and $BC \parallel OM$, with center X on Ω. Consider translating rectangle $ABCD$ along the radius XO to a rectangle $A'B'C'D'$ now centered at O. It is now clear that that every point inside $ABCD$ is a translate of a point in $A'B'C'D'$, and furthermore, any rectangle $ABCD$ translates along the appropriate radius to the same rectangle $A'B'C'D'$.

We see that the boundary of this region can be constructed by constructing a quarter-circle at each vertex, then connecting these quarter-circles with tangents to form four rectangular regions. Now, splitting our region in to four quarter circles and five rectangles, we compute the desired area to be

$$4 \cdot \frac{1}{4}(8)^2\pi + 2(4 \cdot 8) + 2(5 \cdot 8) + (4 \cdot 5) = 164 + 64\pi.$$