4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Guts Round

1. [**5**] Determine the remainder when $1 + 2 + \cdots + 2014$ is divided by 2012.

   Answer: $\boxed{1009}$ We wish to find the value of $1 + 2 + \cdots + 2014$ modulo 2012. We have

   $$1 + 2 + \cdots + 2014 = \frac{1}{2}(2014)(2015) = 1007 \cdot 2015 \equiv 1007 \cdot 3 = 3021 \equiv 1009 \pmod{2012}.$$

   Remark: Note that, since 2 is not relatively prime to 2012, that this is not the same as

   $$\frac{1}{2}(2)(3) \equiv 3 \pmod{2012}.$$

2. [**5**] Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 4$. Let E be the point on BC with $BE = 3$, and let F be the point on segment AE such that F lies halfway between the segments AB and CD. If G is the point of intersection of DF and BC, find BG.

   Answer: $\boxed{1}$ Note that since F is a point halfway between AB and AC, the diagram must be symmetric about the line through F parallel to AB. Hence, G must be the reflection of E across the midpoint of BC. Therefore, $BG = EC = 1$.

3. [**5**] Let x be a real number such that $2^x = 3$. Determine the value of 4^{3x+2}.

   Answer: $\boxed{11664}$ We have

   $$4^{3x+2} = 4^{3x} \cdot 4^2 = (2^2)^{3x} \cdot 16 = 2^{6x} \cdot 16 = (2^x)^6 \cdot 16 = 3^6 \cdot 16 = 11664.$$

4. [**6**] Determine which of the following numbers is smallest in value: $54\sqrt{3}, 144, 108\sqrt{6} - 108\sqrt{2}$.

   Answer: $\boxed{54\sqrt{3}}$ We can first compare $54\sqrt{3}$ and 144. Note that $\sqrt{3} < 2$ and $\frac{144}{54} = \frac{8}{3} > 2$. Hence, $54\sqrt{3}$ is less. Now, we wish to compare this to $108\sqrt{6} - 108\sqrt{2}$. This is equivalent to comparing $\sqrt{3}$ to $2(\sqrt{6} - \sqrt{2})$. We claim that $\sqrt{3} < 2(\sqrt{6} - \sqrt{2})$. To prove this, square both sides to get $3 < 4(8 - 4\sqrt{3})$ or $\sqrt{3} < \frac{29}{16}$ which is true because $\frac{29^2}{16^2} = \frac{841}{256} > 3$. We can reverse this sequence of squarings because, at each step, we make sure that both our values are positive after taking the square root. Hence, $54\sqrt{3}$ is the smallest.

5. [**6**] Charlie folds an $\frac{17}{2}$-inch by 11-inch piece of paper in half twice, each time along a straight line parallel to one of the paper's edges. What is the smallest possible perimeter of the piece after two such folds?

   Answer: $\boxed{\frac{39}{2}}$ Note that a piece of paper is folded in half, one pair of opposite sides is preserved and the other pair is halved. Hence, the net effect on the perimeter is to decrease it by one of the side lengths. Hence, the original perimeter is $2\left(\frac{17}{2}\right) + 2 \cdot 11 = 39$ and by considering the cases of folding twice along one edge or folding once along each edge, one can see that this perimeter can be decreased by at most $11 + \frac{17}{2} = \frac{39}{2}$. Hence, the minimal perimeter is $\frac{39}{2}$.

6. [**6**] To survive the coming Cambridge winter, Chim Tu doesn't wear one T-shirt, but instead wears up to FOUR T-shirts, all in different colors. An *outfit* consists of three or more T-shirts, put on one on top of the other in some order, such that two outfits are distinct if the sets of T-shirts used are different or the sets of T-shirts used are the same but the order in which they are worn is different. Given that Chim Tu changes his outfit every three days, and otherwise never wears the same outfit twice, how many days of winter can Chim Tu survive? (Needless to say, he only has four t-shirts.)

Answer: $\boxed{144}$ We note that there are 4 choices for Chim Tu's innermost T-shirt, 3 choices for the next, and 2 choices for the next. At this point, he has exactly 1 T-shirt left, and 2 choices: either he puts that one on as well or he discards it. Thus, he has a total of $4 \times 3 \times 2 \times 2 = 48$ outfits, and can survive for $48 \times 3 = 144$ days.

7. [**7**] How many ordered triples of positive integers (a, b, c) are there for which $a^4 b^2 c = 54000$?

   Answer: $\boxed{16}$ We note that $54000 = 2^4 \times 3^3 \times 5^3$. Hence, we must have $a = 2^{a_1} 3^{a_2} 5^{a_3}$, $b = 2^{b_1} 3^{b_2} 5^{b_3}$, $c = 2^{c_1} 3^{c_2} 5^{c_3}$. We look at each prime factor individually:

   - $4a_1 + 2b_1 + c_1 = 4$ gives 4 solutions: $(1, 0, 0), (0, 2, 0), (0, 1, 2), (0, 0, 4)$
   - $4a_2 + 2b_2 + c_2 = 3$ and $4a_3 + 2b_3 + c_3 = 3$ each give 2 solutions: $(0, 1, 1), (0, 1, 3)$.

   Hence, we have a total of $4 \times 2 \times 2 = 16$ solutions.

8. [**7**] Let a, b, c be not necessarily distinct integers between 1 and 2011, inclusive. Find the smallest possible value of $\dfrac{ab + c}{a + b + c}$.

   Answer: $\boxed{\frac{2}{3}}$ We have

$$\frac{ab + c}{a + b + c} = \frac{ab - a - b}{a + b + c} + 1.$$

   We note that $\frac{ab-a-b}{a+b+c} < 0 \Leftrightarrow (a - 1)(b - 1) < 1$, which only occurs when either $a = 1$ or $b = 1$. Without loss of generality, let $a = 1$. Then, we have a value of

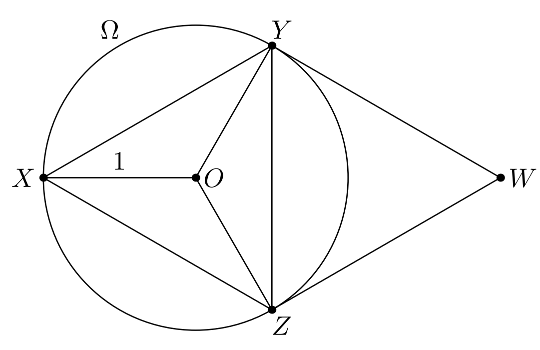
$$\frac{-1}{b + c + a} + 1.$$

   We see that this is minimized when b and c are also minimized (so $b = c = 1$), for a value of $\frac{2}{3}$.

9. [**7**] Unit circle Ω has points X, Y, Z on its circumference so that XYZ is an equilateral triangle. Let W be a point other than X in the plane such that triangle WYZ is also equilateral. Determine the area of the region inside triangle WYZ that lies outside circle Ω.

   Answer: $\boxed{\frac{3\sqrt{3}-\pi}{3}}$ Let O be the center of the circle. Then, we note that since $\angle WYZ = 60^\circ = \angle YXZ$, that YW is tangent to Ω. Similarly, WZ is tangent to Ω. Now, we note that the circular segment corresponding to YZ is equal to $\frac{1}{3}$ the area of Ω less the area of triangle OYZ. Hence, our total area is

$$[WYZ] - \frac{1}{3}[\Omega] + [YOZ] = \frac{3\sqrt{3}}{4} - \frac{1}{3}\pi + \frac{\sqrt{3}}{4} = \frac{3\sqrt{3} - \pi}{3}.$$



10. [**8**] Determine the number of integers D such that whenever a and b are both real numbers with $-1/4 < a, b < 1/4$, then $|a^2 - Db^2| < 1$.

    Answer: $\boxed{32}$ We have

$$-1 < a^2 - Db^2 < 1 \Rightarrow \frac{a^2 - 1}{b^2} < D < \frac{a^2 + 1}{b^2}.$$

We have $\frac{a^2-1}{b^2}$ is maximal at $-15 = \frac{.25^2-1}{.25^2}$ and $\frac{a^2+1}{b^2}$ is minimal at $\frac{0^2+1}{.25^2} = 16$. However, since we cannot have $a, b = \pm.25$, checking border cases of -15 and 16 shows that both of these values are possible for D. Hence, $-15 \leq D \leq 16$, so there are 32 possible values of D.

11. [**8**] For positive integers m, n, let $\gcd(m, n)$ denote the largest positive integer that is a factor of both m and n. Compute

$$\sum_{n=1}^{91} \gcd(n, 91).$$

Answer: $\boxed{325}$ Since $91 = 7 \times 13$, we see that the possible values of $\gcd(n, 91)$ are $1, 7, 13, 91$. For $1 \leq n \leq 91$, there is only one value of n such that $\gcd(n, 91) = 91$. Then, we see that there are 12 values of n for which $\gcd(n, 91) = 7$ (namely, multiples of 7 other than 91), 6 values of n for which $\gcd(n, 91) = 13$ (the multiples of 13 other than 91), and $91 - 1 - 6 - 12 = 72$ values of n for which $\gcd(n, 91) = 1$. Hence, our answer is $1 \times 91 + 12 \times 7 + 6 \times 13 + 72 \times 1 = 325$.

12. [**8**] Joe has written 5 questions of different difficulties for a test with problems numbered 1 though 5. He wants to make sure that problem i is harder than problem j whenever $i - j \geq 3$. In how many ways can he order the problems for his test?

Answer: $\boxed{25}$ We will write $p_i > p_j$ for integers i, j when the ith problem is harder than the jth problem. For the problem conditions to be true, we must have $p_4 > p_1$, $p_5 > p_2$, and $p_5 > p_1$.

Then, out of $5! = 120$ total orderings, we see that in half of them satisfy $p_4 > p_1$ and half satisfy $p_5 > p_2$, and that these two events occur independently. Hence, there are $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(120) = 30$ orderings which satisfy the first two conditions. Then, we see that there are $\frac{4!}{2!2!} = 6$ orderings of p_1, p_2, p_4, p_5 which work; of these, only $p_4 > p_1 > p_5 > p_2$ violates the condition $p_5 > p_1$. Consequently, we have $\frac{5}{6}(30) = 25$ good problem orderings.

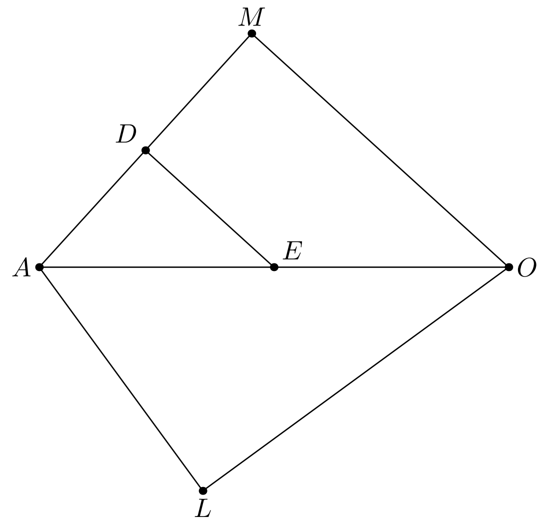
13. [**8**] Tac is dressing his cat to go outside. He has four indistinguishable socks, four indistinguishable shoes, and 4 indistinguishable show-shoes. In a hurry, Tac randomly pulls pieces of clothing out of a door and tries to put them on a random one of his cat's legs; however, Tac never tries to put more than one of each type of clothing on each leg of his cat. What is the probability that, after Tac is done, the snow-shoe on each of his cat's legs is on top of the shoe, which is on top of the sock?

Answer: $\boxed{\frac{1}{1296}}$ On each leg, Tac's cat will get a shoe, a sock, and a snow-shoe in a random order. Thus, the probability that they will be put on in order for any given leg is $\frac{1}{3!} = \frac{1}{6}$. Thus, the probability that this will occur for all 4 legs is $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$.

14. [**8**] Let $AMOL$ be a quadrilateral with $AM = 10$, $MO = 11$, and $OL = 12$. Given that the perpendicular bisectors of sides AM and OL intersect at the midpoint of segment AO, find the length of side LA.

Answer: $\boxed{\sqrt{77}}$ Let D be the midpoint of AM and E be the midpoint of AO. Then, we note that $ADE \sim AMO$, so M is a right angle. Similarly, L is a right angle. Consequently, we get that

$$AO^2 = OM^2 + AM^2 \Rightarrow AL = \sqrt{AO^2 - OL^2} = \sqrt{11^2 + 10^2 - 12^2} = \sqrt{77}.$$



15. [**8**] For positive integers n, let $L(n)$ be the largest factor of n other than n itself. Determine the number of ordered pairs of composite positive integers (m, n) for which $L(m)L(n) = 80$.

Answer: $\boxed{12}$ Let x be an integer, and let p_x be the smallest prime factor of x. Then, if $L(a) = x$, we note that we must have $a = px$ for some prime $p \le p_x$. (Otherwise, if $p > p_x$, then $\frac{px}{p_x} > x$. If p is composite, then $kx > x$ for some factor k of x.)

So we have:

- $L(a) = 2, 4, 8, 10, 16, 20, 40 \Rightarrow 1$ value for a
- $L(a) = 5 \Rightarrow 3$ values for a

Hence, we note that, since m and n are composite, we cannot have $L(m) = 1$ or $L(n) = 1$, so the possible pairs $(L(m), L(n))$ are $(2, 40), (4, 20), (5, 16), (8, 10)$ and vice-versa.

We add the number of choices for each pair, and double since m and n are interchangeable, to get $2(1 \times 1 + 1 \times 1 + 3 \times 1 + 1 \times 1) = 12$ possible ordered pairs (m, n).

16. [**10**] A small fish is holding 17 cards, labeled 1 through 17, which he shuffles into a random order. Then, he notices that although the cards are not currently sorted in ascending order, he can sort them into ascending order by removing one card and putting it back in a *different* position (at the beginning, between some two cards, or at the end). In how many possible orders could his cards currently be?

Answer: $\boxed{256}$ Instead of looking at moves which put the cards in order, we start with the cards in order and consider possible starting positions by backtracking one move: each of 17 cards can be moved to 16 new places. But moving card k between card $k+1$ and card $k+2$ is equivalent to moving card $k+1$ between card $k-1$ and card k. We note that these are the only possible pairs of moves which produce the same result, so we have double counted 16 moves. Thus, we have a total of $17 \times 16 - 16 = 256$ possible initial positions.

17. [**10**] For a positive integer n, let $p(n)$ denote the product of the positive integer factors of n. Determine the number of factors n of 2310 for which $p(n)$ is a perfect square.

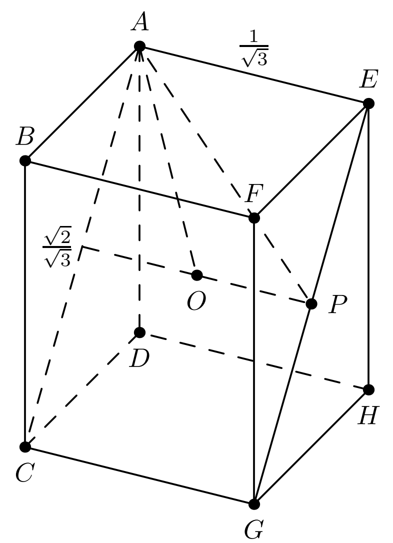
Answer: $\boxed{27}$ Note that $2310 = 2 \times 3 \times 5 \times 7 \times 11$. In general, we see that if n has $d(n)$ positive integer factors, then $p(n) = n^{\frac{d}{2}}$ since we can pair factors $(d, \frac{n}{d})$ which multiply to n. As a result, $p(n)$ is a square if and only if n is a square or d is a multiple of 4.

Thus, because 2310 is not divisible by the square of any prime, we claim that for integers n dividing 2310, $p(n)$ is even if and only if n is not prime. Clearly, $p(n)$ is simply equal to n when n is prime, and $p(1) = 1$, so it suffices to check the case when n is composite. Suppose that $n = p_1 p_2 \cdots p_k$, where $k > 1$ and $\{p_1, \ldots, p_k\}$ is some subset of $\{2, 3, 5, 7, 11\}$. Then, we see that n has 2^k factors, and that $4 \mid 2^k$, so $p(n)$ is a square.

Since 2310 has $2^5 = 32$ factors, five of which are prime, 27 of them have $p(n)$ even.

18. [**10**] Consider a cube $ABCDEFGH$, where $ABCD$ and $EFGH$ are faces, and segments AE, BF, CG, DH are edges of the cube. Let P be the center of face $EFGH$, and let O be the center of the cube. Given that $AG = 1$, determine the area of triangle AOP.

Answer: $\boxed{\frac{\sqrt{2}}{24}}$ From $AG = 1$, we get that $AE = \frac{1}{\sqrt{3}}$ and $AC = \frac{\sqrt{2}}{\sqrt{3}}$. We note that triangle AOP is located in the plane of rectangle $ACGE$. Since $OP \parallel CG$ and O is halway between AC and EG, we get that $[AOP] = \frac{1}{8}[ACGE]$. Hence, $[AOP] = \frac{1}{8}(\frac{1}{\sqrt{3}})(\frac{\sqrt{2}}{\sqrt{3}}) = \frac{\sqrt{2}}{24}$.



19. [**10**] Let $ABCD$ be a rectangle with $AB = 3$ and $BC = 7$. Let W be a point on segment AB such that $AW = 1$. Let X, Y, Z be points on segments BC, CD, DA, respectively, so that quadrilateral $WXYZ$ is a rectangle, and $BX < XC$. Determine the length of segment BX.

Answer: $\boxed{\frac{7-\sqrt{41}}{2}}$ We note that

$$\angle YXC = 90 - \angle WXB = \angle XWB = 90 - \angle AWZ = \angle AZW$$

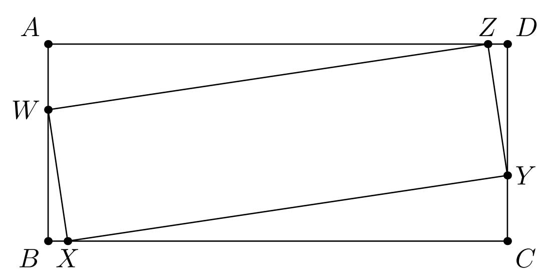
gives us that $XYC \cong ZWA$ and $XYZ \sim WXB$. Consequently, we get that $YC = AW = 1$. From $XYZ \sim WXB$, we get that

$$\frac{BX}{BW} = \frac{CY}{CX} \Rightarrow \frac{BX}{2} = \frac{1}{7 - BX}$$

from which we get

$$BX^2 - 7BX + 2 = 0 \Rightarrow BX = \frac{7 - \sqrt{41}}{2}$$

(since we have $BX < CX$).



20. [**10**] The UEFA Champions League playoffs is a 16-team soccer tournament in which Spanish teams always win against non-Spanish teams. In each of 4 rounds, each remaining team is *randomly* paired against one other team; the winner advances to the next round, and the loser is permanently knocked out of the tournament. If 3 of the 16 teams are Spanish, what is the probability that there are 2 Spanish teams in the final round?

Answer: $\boxed{\frac{4}{5}}$ We note that the probability there are not two Spanish teams in the final two is the probability that the 3 of them have already competed against each other in previous rounds. Note that the random pairings in each round is equivalent, by the final round, to dividing the 16 into two groups of 8 and taking a winner from each. Now, letting the Spanish teams be A, B, and C, once we fix the group in which A is contained, the probability that B is contained in this group as well is $7/15$. Likewise, the probability that C will be in the same group as A and B is now $6/14$. Our answer is thus

$$1-\left(\frac{7}{15}\right)\left(\frac{6}{14}\right)=\frac{4}{5}.$$

21. [**10**] Let $P(x)=x^4+2x^3-13x^2-14x+24$ be a polynomial with roots r_1,r_2,r_3,r_4. Let Q be the quartic polynomial with roots r_1^2,r_2^2,r_3^2,r_4^2, such that the coefficient of the x^4 term of Q is 1. Simplify the quotient $Q(x^2)/P(x)$, leaving your answer in terms of x. (You may assume that x is not equal to any of r_1,r_2,r_3,r_4).

Answer: $\boxed{x^4-2x^3-13x^2+14x+24}$ We note that we must have

$$Q(x)=(x-r_1^2)(x-r_2^2)(x-r_3^2)(x-r_4^2)\Rightarrow Q(x^2)=(x^2-r_1^2)(x^2-r_2^2)(x^2-r_3^2)(x^2-r_4^2)$$

. Since $P(x)=(x-r_1)(x-r_2)(x-r_3)(x-r_4)$, we get that

$$Q(x^2)/P(x)=(x+r_1)(x+r_2)(x+r_3)(x+r_4).$$

Thus, $Q(x^2)/P(x)=(-1)^4P(-x)=P(-x)$, so it follows that

$$Q(x^2)/P(x)=x^4-2x^3-13x^2+14x+24.$$

22. [**12**] Let ABC be a triangle with $AB=23$, $BC=24$, and $CA=27$. Let D be the point on segment AC such that the incircles of triangles BAD and BCD are tangent. Determine the ratio CD/DA.

Answer: $\boxed{\frac{14}{13}}$ Let X,Z,E be the points of tangency of the incircle of ABD to AB,BD,DA respectively. Let Y,Z,F be the points of tangency of the incircle of CBD to CB,BD,DC respectively. We note that

$$CB+BD+DC=CY+YB+BZ+ZD+DF+FC=2(CY)+2(BY)+2(DF)2(24)+2(DF)$$

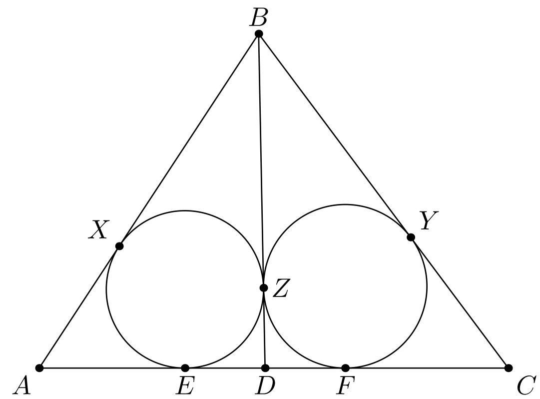
by equal tangents, and that similarly

$$AB+BD+DA=2(23)+2(DE).$$

Since $DE=DZ=DF$ by equal tangents, we can subtract the equations above to get that

$$CB+CD-AB-AD=2(24)-24(23)\Rightarrow CD-DA=1.$$

Since we know that $CD+DA=27$, we get that $CD=14$, $DA=13$, so the desired ratio is $\frac{14}{13}$.



23. [**12**] Let $N = \overline{5AB37C2}$, where A, B, C are digits between 0 and 9, inclusive, and N is a 7-digit positive integer. If N is divisible by 792, determine all possible ordered triples (A, B, C).

Answer: $\boxed{(0,5,5),(4,5,1),(6,4,9)}$ First, note that $792 = 2^3 \times 3^2 \times 11$. So we get that

$$8 \mid N \Rightarrow 8 \mid \overline{7C2} \Rightarrow 8 \mid 10C + 6 \Rightarrow C = 1, 5, 9$$

$$9 \mid N \Rightarrow 9 \mid 5 + A + B + 3 + 7 + C + 2 \Rightarrow A + B + C = 1, 10, 19$$

$$11 \mid N \Rightarrow 11 \mid 5 - A + B - 3 + 7 - C + 2 \Rightarrow -A + B - C = -11, 0$$

Adding the last two equations, and noting that they sum to $2B$, which must be even, we get that $B = 4, 5$.

Checking values of C we get possible triplets of $(0, 5, 5)$, $(4, 5, 1)$, and $(6, 4, 9)$.

24. [**12**] Three not necessarily distinct positive integers between 1 and 99, inclusive, are written in a row on a blackboard. Then, the numbers, without including any leading zeros, are concatenated to form a new integer N. For example, if the integers written, in order, are 25, 6, and 12, then $N = 25612$ (and not $N = 250612$). Determine the number of possible values of N.

Answer: $\boxed{825957}$ We will divide this into cases based on the number of digits of N.

- *Case 1:* 6 digits. Then each of the three numbers must have two digits, so we have 90 choices for each. So we have a total of $90^3 = 729000$ possibilities.
- *Case 2:* 5 digits. Then, exactly one of the three numbers is between 1 and 9, inclusive. We consider cases on the presence of 0s in N.
  - No 0s. Then, we have 9 choices for each digit, for a total of $9^5 = 59049$ choices.
  - One 0. Then, the 0 can be the second, third, fourth, or fifth digit, and 9 choices for each of the other 4 digits. Then, we have a total of $4 \times 9^4 = 26244$ choices.
  - Two 0s. Then, there must be at least one digit between them and they cannot be in the first digit, giving us 3 choices for the positioning of the 0s. Then, we have a total of $3 * 9^3 = 2187$ choices.

  So we have a total of $59049 + 26244 + 2187 = 87480$ choices in this case.
- *Case 3:* 4 digits. Again, we casework on the presence of 0s.
  - No 0s. Then, we have $9^4 = 6561$ choices.
  - One 0. Then, the 0 can go in the second, third, or fourth digit, so we have $3 \times 9^3 = 2187$ choices.

  So we have a total of $6561 + 2187 = 8748$ choices in this case.
- *Case 4:* 3 digits. Then, we cannot have any 0s, so we have a total of $9^3 = 729$ choices.

Hence, we have a total of $729000 + 87480 + 8748 + 729 = 825957$ choices for N.

25. [**12**] Let XYZ be an equilateral triangle, and let K, L, M be points on sides XY, YZ, ZX, respectively, such that $XK/KY = B$, $YL/LZ = 1/C$, and $ZM/MX = 1$. Determine the ratio of the area of triangle KLM to the area of triangle XYZ.

Answer: $\boxed{\frac{1}{5}}$ First, we note that

$$[KLM] = [XYZ] - [XKM] - [YLK] - [ZML].$$

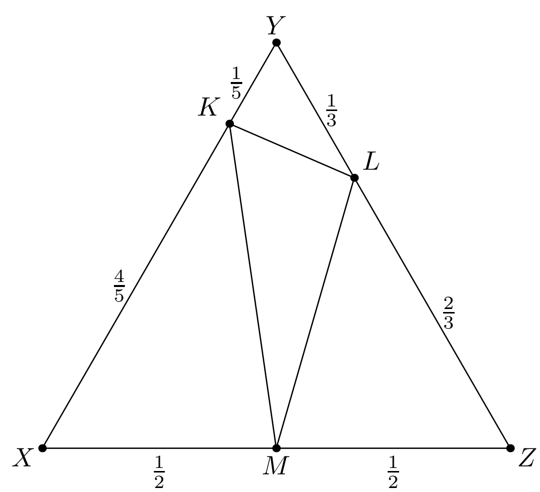
Then, note that

$$[XKM] = \frac{XK}{XY} \cdot \frac{XM}{XZ} \cdot [XYZ] = \frac{B}{B+1} \cdot \frac{1}{2} \cdot [XYZ]$$
$$[YLK] = \frac{YL}{YZ} \cdot \frac{YK}{YX} \cdot [XYZ] = \frac{1}{C+1} \cdot \frac{1}{B+1} \cdot [XYZ]$$
$$[ZML] = \frac{ZM}{ZX} \cdot \frac{ZL}{ZY} \cdot [XYZ] = \frac{1}{2} \cdot \frac{1}{C+1} \cdot [XYZ]$$

Consequently,

$$\begin{aligned} A &= \frac{[KLM]}{[XYZ]} \\ &= 1 - \frac{B}{B+1} \cdot \frac{1}{2} - \frac{1}{C+1} \cdot \frac{1}{B+1} - \frac{C}{C+1} \cdot \frac{1}{2} \\ &= \frac{B+C}{(B+1)(C+1)(2)} \end{aligned}$$

If we solve our system of equations for A, B, C, we get that $A = \frac{1}{5}$.



26. [**12**] Determine the positive real value of x for which

$$\sqrt{2 + AC + 2Cx} + \sqrt{AC - 2 + 2Ax} = \sqrt{2(A+C)x + 2AC}.$$

Answer: $\boxed{4}$ Note that if we have $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ for non-negative reals a, b, then squaring gives us that $2\sqrt{ab} = 0$, so that either $a = 0$ or $b = 0$.

Now, note that

$$(2 + AC + 2Cx) + (AC - 2 + 2Ax) = (2(A+C)x + 2AC).$$

Consequently, either $(2 + AC + 2Cx)$ or $(AC - 2 + 2Ax)$ must be equal to 0. However, we observe from the problems that both A, C, and x must be non-negative, so $(2 + AC + 2Cx) > 0$. As a result, we know that $AC - 2 + 2Ax = 0$, or that

$$B = x = \frac{2 - AC}{2A}.$$

If we solve our system of equations for A, B, C, we get that $B = 4$.

27. [**12**] In-Young generates a string of B zeroes and ones using the following method:

- First, she flips a fair coin. If it lands heads, her first digit will be a 0, and if it lands tails, her first digit will be a 1.
- For each subsequent bit, she flips an unfair coin, which lands heads with probability A. If the coin lands heads, she writes down the number (zero or one) different from previous digit, while if the coin lands tails, she writes down the previous digit again.

What is the expected value of the number of zeroes in her string?

Answer: $\boxed{2}$ Since each digit is dependent on the previous, and the first digit is random, we note that the probability that In Young obtains a particular string is the same probability as that she obtains the inverse string (i.e. that where the positions of the 0s and 1s are swapped). Consequently, we would expect that half of her digits are 0s, so that

$$C = \frac{B}{2}.$$

If we solve our system of equations for A, B, C, we get that $C = 2$.

Solution of the system of equations for Problems 25, 26, 27:

Thus, we have the three equations

$$A = \frac{B + C}{(B + 1)(C + 1)(2)}, \quad B = \frac{2 - AC}{2A}, \quad C = \frac{B}{2}$$

Plugging the last equation into the first two results in

$$A = \frac{3B}{(B + 1)(B + 2)(2)} \Rightarrow B = \frac{4 - AB}{4A}$$

Rearranging the second equation gives

$$4AB = 4 - AB \Rightarrow AB = \frac{4}{5} \Rightarrow A = \frac{4}{5B}$$

Then, plugging this into the first equation gives

$$\frac{4}{5B} = \frac{3B}{(B + 1)(B + 2)(2)}$$
$$\Rightarrow 15B^2 = 8B^2 + 24B + 16$$
$$\Rightarrow 7B^2 - 24B - 16 = 0$$
$$\Rightarrow (7B + 4)(B - 4) = 0$$

Since we know that $B > 0$, we get that $B = 4$. Plugging this back in gives $A = \frac{1}{5}$ and $C = 2$.

28. [**14**] Determine the value of

$$\sum_{k=1}^{2011} \frac{k-1}{k!(2011-k)!}.$$

Answer: $\boxed{\frac{2009(2^{2010})+1}{2011!}}$ We note that

$$\begin{aligned}
(2011!)\sum_{k=1}^{2011} \frac{k-1}{k!(2011-k)!} &= \sum_{k=1}^{2011} \frac{(2011!)(k-1)}{k!(2011-k)!} \\
&= \sum_{k=1}^{2011} \frac{k(2011)!}{k!(2011-k)!} - \sum_{k=1}^{2011} \frac{2011!}{k!(2011-k)!} \\
&= \sum_{k=1}^{2011} k\binom{2011}{k} - \sum_{k=1}^{2011} \binom{2011}{k} \\
&= (2011)(2^{2010}) - (2^{2011}-1)
\end{aligned}$$

Thus, we get an answer of $\left(2009\left(2^{2010}\right)+1\right)/(2011!)$.

Note: To compute the last two sums, observe that

$$\sum_{k=0}^{2011} \binom{2011}{k} = (1+1)^{2011} = 2^{2011}$$

by the Binomial Theorem, and that

$$\sum_{k=0}^{2011} k\binom{2011}{k} = \frac{1}{2}\left(\sum_{k=0}^{2011} k\binom{2011}{k} + \sum_{k=0}^{2011}(2011-k)\binom{2011}{2011-k}\right) = 2011\left(2^{2010}\right).$$

29. [**14**] Let ABC be a triangle with $AB=4$, $BC=8$, and $CA=5$. Let M be the midpoint of BC, and let D be the point on the circumcircle of ABC so that segment AD intersects the interior of ABC, and $\angle BAD = \angle CAM$. Let AD intersect side BC at X. Compute the ratio AX/AD.

Answer: $\boxed{\frac{9}{41}}$ Let E be the intersection of AM with the circumcircle of ABC. We note that, by equal angles $ADC \sim ABM$, so that

$$AD = AC\left(\frac{AB}{AM}\right) = \frac{20}{AM}.$$

Using the law of cosines on ABC, we get that

$$\cos B = \frac{4^2+8^2-5^2}{2(4)(8)} = \frac{55}{64}.$$

Then, using the law of cosines on ABM, we get that

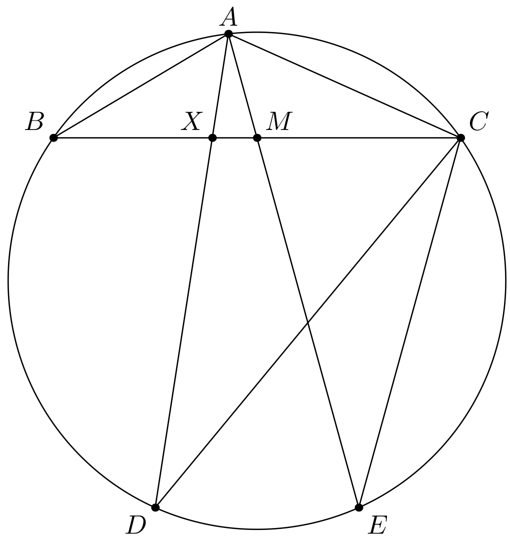
$$AM = \sqrt{4^2+4^2-2(4)(4)\cos B} = \frac{3}{\sqrt{2}} \Rightarrow AD = \frac{20\sqrt{2}}{3}.$$

Applying Power of a Point on M,

$$(AM)(ME) = (BM)(MC) \Rightarrow ME = \frac{16\sqrt{2}}{3} \Rightarrow AE = \frac{41\sqrt{2}}{6}.$$

Then, we note that $AXB \sim ACE$, so that

$$AX = AB\left(\frac{AC}{AE}\right) = \frac{60\sqrt{2}}{41} \Rightarrow \frac{AX}{AD} = \frac{9}{41}$$



30. [**14**] Let S be a set of consecutive positive integers such that for any integer n in S, the sum of the digits of n is not a multiple of 11. Determine the largest possible number of elements of S.

    Answer: $\boxed{38}$ We claim that the answer is 38. This can be achieved by taking the smallest integer in the set to be 999981. Then, our sums of digits of the integers in the set are

    $$45, \ldots, 53, 45, \ldots, 54, 1, \ldots, 10, 2, \ldots, 10,$$

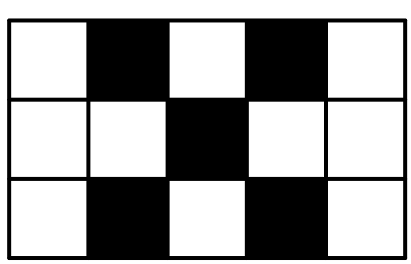
    none of which are divisible by 11.

    Suppose now that we can find a larger set S: then we can then take a 39-element subset of S which has the same property. Note that this implies that there are consecutive integers $a-1, a, a+1$ for which $10b, \ldots, 10b+9$ are all in S for $b = a-1, a, a+1$. Now, let $10a$ have sum of digits N. Then, the sums of digits of $10a+1, 10a+2, \ldots, 10a+9$ are $N+1, N+2, \ldots, N+9$, respectively, and it follows that $n \equiv 1 \pmod{11}$.

    If the tens digit of $10a$ is not 9, note that $10(a+1)+9$ has sum of digits $N+10$, which is divisible by 11, a contradiction. On the other hand, if the tens digit of $10a$ is 9, the sum of digits of $10(a-1)$ is $N-1$, which is also divisible by 11. Thus, S has at most 38 elements.

    Motivation: We want to focus on subsets of S of the form $\{10a, \ldots, 10a+9\}$, since the sum of digits goes up by 1 most of the time. If the tens digit of $10a$ is anything other than 0 or 9, we see that S can at most contain the integers between $10a-8$ and $10a+18$, inclusive. However, we can attempt to make $10(a-1)+9$ have sum of digits congruent to $N+9$ modulo 11, as to be able to add as many integers to the beginning as possible, which can be achieved by making $10(a-1)+9$ end in the appropriate number of nines. We see that we want to take $10(a-1)+9 = 999999$ so that the sum of digits upon adding 1 goes down by $53 \equiv 9 \pmod{11}$, giving the example we constructed previously.

31. [**17**] Each square in a 3×10 grid is colored black or white. Let N be the number of ways this can be done in such a way that no five squares in an 'X' configuration (as shown by the black squares below) are all white or all black. Determine $\sqrt{N}$.



Answer: $\boxed{25636}$ Note that we may label half of the cells in our board the number 0 and the other half 1, in such a way that squares labeled 0 are adjacent only to squares labeled 1 and vice versa. In other words, we make this labeling in a 'checkerboard' pattern. Since cells in an 'X' formation are all labeled with the same number, the number of ways to color the cells labeled 0 is $\sqrt{N}$, and the same is true of coloring the cells labeled 1.

Let a_{2n} be the number of ways to color the squares labeled 0 in a 3 by $2n$ grid without a monochromatic 'X' formation; we want to find a_{10}. Without loss of generality, let the rightmost column of our grid have two cells labeled 0. Let b_{2n} be the number of such colorings on a 3 by $2n$ grid which do not have two black squares in the rightmost column and do not contain a monochromatic 'X', which we note is also the number of such colorings which do not have two white squares in the rightmost column.

Now, we will establish a recursion on a_{2n} and b_{2n}.We have two cases:

- *Case 1:* All three squares in the last two columns are the same color. For a_{2n}, there are 2 ways to color these last three squares, and for b_{2n} there is 1 way to color them. Then, we see that there are b_{2n-2} ways to color the remaining $2n-2$ columns.
- *Case 2:* The last three squares are not all the same color. For a_{2n}, there are 6 ways to color the last three squares, and for b_{2n} there are 5 ways to color them. Then, there are a_{2n-2} ways to color the remaining $2n-2$ columns.

Consequently, we get the recursions $a_{2n} = 6a_{2n-2} + 2b_{2n-2}$ and $b_{2n} = 5a_{2n-2} + b_{2n-2}$. From the first equation, we get that $b_{2n} = \frac{1}{2}a_{2n+2} - 3a_{2n}$. Plugging this in to the second equations results in the recurion

$$\frac{1}{2}a_{2n+2} - 3a_{2n} = 5a_{2n-2} + \frac{1}{2}a_{2n} - 3a_{2n-2} \Rightarrow a_{2n+2} = 7a_{2n} + 4a_{2n-2}.$$

Now, we can easily see that $a_0 = 1$ and $a_2 = 2^3 = 8$, so we compute $a_{10} = 25636$.

32. [**17**] Find all real numbers x satisfying

$$x^9 + \frac{9}{8}x^6 + \frac{27}{64}x^3 - x + \frac{219}{512} = 0.$$

Answer: $\boxed{\frac{1}{2}, \frac{-1\pm\sqrt{13}}{4}}$ Note that we can re-write the given equation as

$$\sqrt[3]{x - \frac{3}{8}} = x^3 + \frac{3}{8}.$$

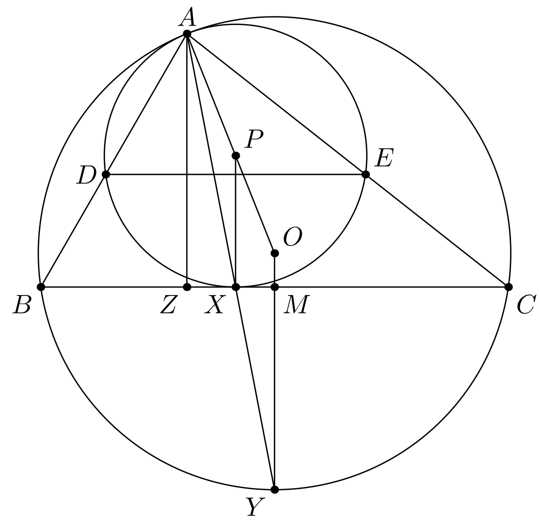
Furthermore, the functions of x on either side, we see, are inverses of each other and increasing. Let $f(x) = \sqrt[3]{x - \frac{3}{8}}$. Suppose that $f(x) = y = f^{-1}(x)$. Then, $f(y) = x$. However, if $x < y$, we have $f(x) > f(y)$, contradicting the fact that f is increasing, and similarly, if $y < x$, we have $f(x) < f(y)$, again a contradiction. Therefore, if $f(x) = f^{-1}(x)$ and both are increasing functions in x, we require $f(x) = x$. This gives the cubic

$$x^3 - x + \frac{3}{8} = 0 \rightarrow \left(x - \frac{1}{2}\right)\left(x^2 + \frac{1}{2}x - \frac{3}{4}\right) = 0,$$

giving $x = \dfrac{1}{2}, \dfrac{-1 \pm \sqrt{13}}{4}$.

33. [**17**] Let ABC be a triangle with $AB = 5$, $BC = 8$, and $CA = 7$. Let Γ be a circle internally tangent to the circumcircle of ABC at A which is also tangent to segment BC. Γ intersects AB and AC at points D and E, respectively. Determine the length of segment DE.

Answer: $\boxed{\frac{40}{9}}$



First, note that a homothety h centered at A takes Γ to the circumcircle of ABC, D to B and E to C, since the two circles are tangent. As a result, we have $DE \parallel BC$. Now, let P be the center of Γ and O be the circumcenter of ABC: by the homothety h, we have $DE/BC = AP/AO$.

Let Γ be tangent to BC at X, and let ray $\overrightarrow{AX}$ meet the circumcircle of ABC at Y. Note that Y is the image of X under h. Furthermore, h takes BC to the tangent line l to the circumcircle of ABC at Y, and since $BC \parallel l$, we must have that Y is the midpoint of arc $\widehat{BC}$. Therefore, AX bisects $\angle BAC$.

Now, let Z be the foot of the altitude from A to BC, and let M be the midpoint of BC, so that $OM \perp BC$. Note that $AP/AO = ZX/ZM$. Now, letting $BC = a = 8$, $CA = b = 7$, and $AB = c = 5$, we compute

$$BZ = c\cos B = \frac{c^2 + a^2 - b^2}{2a} = \frac{5}{2}$$

by the Law of Cosines,

$$BX = \frac{ac}{b+c} = \frac{10}{3}$$

by the Angle Bisector Theorem, and

$$BM = 4.$$

To finish,

$$DE = \frac{(AP)(BC)}{AO} = \frac{(ZX)(BC)}{ZM} = \frac{(5/6)(8)}{(3/2)} = \frac{40}{9}$$

.

34. [**20**] The integer 843301 is prime. The *primorial* of a prime number p, denoted $p\#$, is defined to be the product of all prime numbers less than or equal to p. Determine the number of digits in $843301\#$. Your score will be

$$\max\left\{\left\lfloor 60\left(\frac{1}{3} - \left|\ln\left(\frac{A}{d}\right)\right|\right)\right\rfloor, 0\right\},$$

where A is your answer and d is the actual answer.

Answer: $\boxed{365851}$ **Remark:** $843301\# - 1$ is the largest known prime number of the form $p\# - 1$, where p is prime.

35. [**20**] Let G be the number of Google hits of "guts round" at 10:31PM on October 31, 2011. Let B be the number of Bing hits of "guts round" at the same time. Determine B/G. Your score will be

$$\max\left(0, \left\lfloor 20\left(1 - \frac{20|a-k|}{k}\right)\right\rfloor\right),$$

where k is the actual answer and a is your answer.

Answer: $\boxed{.82721}$ The number of Google hits was 7350. The number of Bing hits was 6080. The answer is thus $6080/7350 = .82721$.

36. [**20**] Order any subset of the following twentieth century mathematical achievements chronologically, from earliest to most recent. If you correctly place at least six of the events in order, your score will be $2(n-5)$, where n is the number of events in your sequence; otherwise, your score will be zero. Note: if you order any number of events with one error, your score will be zero.

A). Axioms for Set Theory published by Zermelo
B). Category Theory introduced by Mac Lane and Eilenberg
C). Collatz Conjecture proposed
D). Erdos number defined by Goffman
E). First United States delegation sent to International Mathematical Olympiad
F). Four Color Theorem proven with computer assistance by Appel and Haken
G). Harvard-MIT Math Tournament founded
H). Hierarchy of grammars described by Chomsky
I). Hilbert Problems stated
J). Incompleteness Theorems published by Godel
K). Million dollar prize for Millennium Problems offered by Clay Mathematics Institute
L). Minimum number of shuffles needed to randomize a deck of cards established by Diaconis
M). Nash Equilibrium introduced in doctoral dissertation
N). Proof of Fermat's Last Theorem completed by Wiles
O). Quicksort algorithm invented by Hoare

Write your answer as a list of letters, without any commas or parentheses.

Answer: $\boxed{\text{IAJCBMHODEFLNGK}}$ The dates are as follows:

A). Axioms for Set Theory published by Zermelo 1908
B). Category Theory introduced by Mac Lane and Eilenberg 1942-1945
C). Collatz Conjecture proposed 1937
D). Erdos number defined by Goffman 1969
E). First United States delegation sent to International Mathematical Olympiad 1974
F). Four Color Theorem proven with computer assistance by Appel and Haken 1976
G). Harvard-MIT Math Tournament founded 1998
H). Hierarchy of grammars described by Chomsky 1956
I). Hilbert Problems stated 1900
J). Incompleteness Theorems published by Godel 1931
K). Million dollar prize for Millennium Problems offered by Clay Mathematics Institute 2000
L). Minimum number of shuffles needed to randomize a deck of cards established by Diaconis 1992
M). Nash Equilibrium introduced in doctoral dissertation 1950
N). Proof of Fermat's Last Theorem completed by Wiles 1994
O). Quicksort algorithm invented by Hoare 1960

so the answer is $IAJCBMHODEFLNGK$.