4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

General Test

1. [**3**] Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$.

   Answer: $\boxed{(1,3),(3,\frac{1}{3})}$ Multiplying the second equation by x gives

   $$x^2 + x^2y = 4x,$$

   and substituting our known value of x^2y gives the quadratic

   $$x^2 - 4x + 3 = 0,$$

   so $x = 1$ or $x = 3$. Hence, we obtain the solutions $(x, y) = (1, 3), (3, 1/3)$.

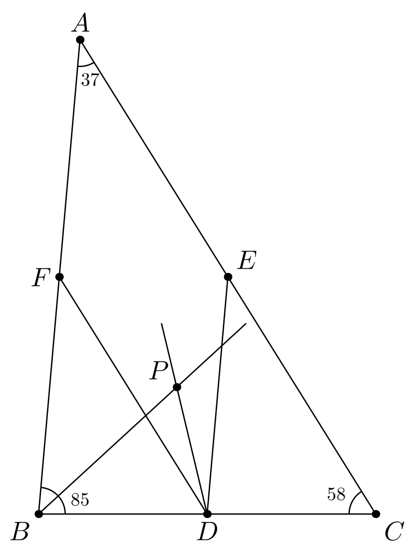
2. [**3**] Let ABC be a triangle, and let D, E, and F be the midpoints of sides BC, CA, and AB, respectively. Let the angle bisectors of $\angle FDE$ and $\angle FBD$ meet at P. Given that $\angle BAC = 37°$ and $\angle CBA = 85°$, determine the degree measure of $\angle BPD$.

   Answer: $\boxed{61°}$ Because D, E, F are midpoints, we have $ABC \sim DEF$. Furthermore, we know that $FD \parallel AC$ and $DE \parallel AB$, so we have

   $$\angle BDF = \angle BCA = 180 - 37 - 85 = 58°.$$

   Also, $\angle FDE = \angle BAC = 37°$. Hence, we have

   $$\angle BPD = 180° - \angle PBD - \angle PDB = 180° - \frac{85°}{2} - \left(\frac{37°}{2} + 58°\right) = 61°.$$



3. [**4**] Alberto, Bernardo, and Carlos are collectively listening to three different songs. Each is simultaneously listening to exactly two songs, and each song is being listened to by exactly two people. In how many ways can this occur?

   Answer: $\boxed{6}$ We have $\binom{3}{2} = 3$ choices for the songs that Alberto is listening to. Then, Bernardo and Carlos must both be listening to the third song. Thus, there are 2 choices for the song that Bernardo shares with Alberto. From here, we see that the songs that everyone is listening to are forced. Thus, there are a total of $3 \times 2 = 6$ ways for the three to be listening to songs.

4. [**4**] Determine the remainder when

$$2^{\frac{1\cdot 2}{2}} + 2^{\frac{2\cdot 3}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}}$$

is divided by 7.

Answer: $\boxed{1}$ We have that $2^3 \equiv 1 \pmod 7$. Hence, it suffices to consider the exponents modulo 3. We note that the exponents are the triangular number and upon division by 3 give the pattern of remainders $1, 0, 0, 1, 0, 0, \ldots$, so what we want is

$$\begin{aligned} 2^{\frac{1\cdot 2}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}} &\equiv 2^1 + 2^0 + 2^0 + 2^1 + \ldots + 2^0 + 2^1 \pmod 7 \\ &\equiv \frac{2010}{3}(2^1 + 2^0 + 2^0) + 2^1 \\ &\equiv (670)(4) + 2 \\ &\equiv 1. \end{aligned}$$

5. [**5**] Find all real values of x for which

$$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} = \frac{1}{4}.$$

Answer: $\boxed{\frac{257}{16}}$ We note that

$$\begin{aligned} \frac{1}{4} &= \frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} \\ &= \frac{\sqrt{x} - \sqrt{x-2}}{(\sqrt{x} + \sqrt{x-2})(\sqrt{x} - \sqrt{x-2})} + \frac{\sqrt{x+2} - \sqrt{x}}{(\sqrt{x+2} + \sqrt{x})(\sqrt{x+2} - \sqrt{x})} \\ &= \frac{\sqrt{x} - \sqrt{x-2}}{2} + \frac{\sqrt{x+2} - \sqrt{x}}{2} \\ &= \frac{1}{2}(\sqrt{x+2} - \sqrt{x-2}), \end{aligned}$$

so that

$$2\sqrt{x+2} - 2\sqrt{x-2} = 1.$$

Squaring, we get that

$$8x - 8\sqrt{(x+2)(x-2)} = 1 \Rightarrow 8x - 1 = 8\sqrt{(x+2)(x-2)}.$$

Squaring again gives

$$64x^2 - 16x + 1 = 64x^2 - 256,$$

so we get that $x = \frac{257}{16}$.

6. [**5**] Five people of heights 65, 66, 67, 68, and 69 inches stand facing forwards in a line. How many orders are there for them to line up, if no person can stand immediately before or after someone who is exactly 1 inch taller or exactly 1 inch shorter than himself?

Answer: $\boxed{14}$ Let the people be A, B, C, D, E so that their heights are in that order, with A the tallest and E the shortest. We will do casework based on the position of C.

- *Case 1:* C is in the middle. Then, B must be on one of the two ends, for two choices. This leaves only one choice for D–the other end. Then, we know the positions of A and E since A cannot neighbor B and E cannot neighbor D. So we have 2 options for this case.

- *Case 2:* C is in the second or fourth spot. Then, we have two choices for the position of C. Without loss of generality, let C be in the second spot. Then, the first and third spots must be A and E, giving us two options. This fixes the positions of B and D, so we have a total of $2 \times 2 = 4$ options for this case.
- *Case 3:* C is in the first or last spot. Then, we have two choices for the position of C. Without loss of generality, let it be in the first spot. Either A or E is in the second spot, giving us two choices. Without loss of generality, let it be A. Then, if D is in the third spot, the positions of B and E are fixed. If E is in third spot, the positions of B and D are fixed, so we have a total of $2 \times 2 \times (1 + 1) = 8$ options for this case.

Hence, we have a total of $2 + 4 + 8 = 14$ possibilities.

7. [**5**] Determine the number of angles θ between 0 and 2π, other than integer multiples of $\pi/2$, such that the quantities $\sin\theta$, $\cos\theta$, and $\tan\theta$ form a geometric sequence in some order.

Answer: $\boxed{4}$ If $\sin\theta, \cos\theta$, and $\tan\theta$ are in a geometric progression, then the product of two must equal the square of the third. Using this criterion, we have 3 cases.

- *Case 1:* $\sin\theta \cdot \tan\theta = \cos^2\theta$. This implies that $(\sin^2\theta) = (\cos^3\theta)$. Writing $\sin^2\theta$ as $1 - \cos^2\theta$ and letting $\cos\theta = x$, we have that $x^3 + x^2 - 1 = 0$. We wish to find the number of solutions of this where $|x| \le 1$. Clearly -1 is not a root. If $-1 < x \le 0$, we have that $x^2 + x^3 \le x^2 < 1$ so $x^3 + x^2 - 1 < 0$ and there are no roots. If $0 < x \le 1$, then $x^3 + x^2 - 1$ is a strictly increasing function. Since it has value -1 at $x = 0$ and value 1 and $x = 1$, there is exactly one root between 0 and 1, non-inclusive. There are 2 values of θ such that $\cos\theta$ equals this root, and thus, two solutions in this case.
- *Case 2:* $\sin\theta \cdot \cos\theta = \tan^2\theta$. This implies that $\cos^3\theta = \sin\theta$. To find the number of solutions in this case, we can analyze the graphs of the functions in different ranges. Note that from $\theta = 0$ to $\theta = \frac{\pi}{2}$, $\cos^3\theta$ decreases strictly from 1 to 0 while $\sin\theta$ increases strictly from 0 to 1. Hence, there is one solution in this range. By a similar argument, a solution exists between $\theta = \pi$ and $\theta = \frac{3\pi}{2}$. In the intervals $[\frac{\pi}{2}, \pi]$ and $[\frac{3\pi}{2}, 2\pi]$, we have that one function is negative and the other is positive, so there are no solutions. Thus, there are two solutions in this case.
- *Case 3:* $\cos\theta \cdot \tan\theta = \sin^2\theta$. This implies that $\sin\theta = \sin^2\theta$, so $\sin\theta = 0, 1$. Clearly the only solutions of these have θ as an integer multiple of $\frac{\pi}{2}$. Thus, there are no pertinent solutions int his case.

We can see that the solutions for the first two cases are mutually exclusive. Hence, there are 4 solutions in total.

8. [**6**] Find the number of integers x such that the following three conditions all hold:

- x is a multiple of 5
- $121 < x < 1331$
- When x is written as an integer in base 11 with no leading 0s (i.e. no 0s at the very left), its rightmost digit is strictly greater than its leftmost digit.

Answer: $\boxed{99}$ We will work in base 11, so let $x = \overline{def}_{11}$ such that $d > 0$. Then, based on the first two conditions, we aim to find multiples of 5 between 100_{11} and 1000_{11}. We note that

$$\overline{def}_{11} \equiv 11^2 \cdot d + 11 \cdot e + f \equiv d + e + f \pmod 5.$$

Hence, x a multiple of 5 if and only if the sum of its digits is a multiple of 5. Thus, we wish to find triples (d, e, f) with elements in $0, 1, 2, \cdots, 9, 10$ such that $d + e + f \equiv 0 \pmod 5$ and $0 < d < f$.

Note that if we choose d and f such that $d < f$, there is exactly one value of e modulo 5 that would make $d + e + f \equiv 0 \pmod 5$. Once the this value of e is fixed, then there are two possibilities for e unless $e \equiv 0 \pmod 5$, in which case there are three possibilities. Thus, our answer is twice the number of ways to choose d and f such that $0 < d < f$ plus the number of ways to choose d and f such that

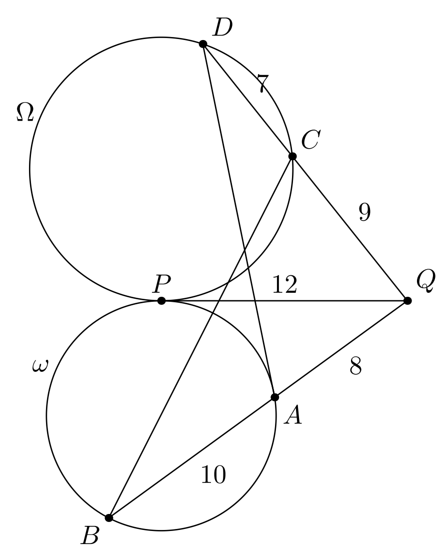
$d+f \equiv 0 \pmod 5$ and $0 < d < f$ (to account for the extra choice for the value of e). Note that the number of ways to choose $0 < d < f$ is just $\binom{10}{2}$ since any any choice of two digits yields exactly one way to order them. The number of ways to choose $d+f \equiv 0 \pmod 5$ and $0 < d < f$ can be found by listing: $(d,f) = (1,4),(1,9),(2,3),(2,8),(3,7),(4,6),(5,10),(6,9),(7,8)$, for 9 such pairings.

Hence, the total is $2\binom{10}{2} + 9 = 99$ possibilities for x.

9. [**7**] Let P and Q be points on line l with $PQ = 12$. Two circles, ω and Ω, are both tangent to l at P and are externally tangent to each other. A line through Q intersects ω at A and B, with A closer to Q than B, such that $AB = 10$. Similarly, another line through Q intersects Ω at C and D, with C closer to Q than D, such that $CD = 7$. Find the ratio AD/BC.

Answer: $\boxed{\frac{8}{9}}$ We first apply the Power of a Point theorem repeatedly. Note that $QA \cdot QB = QP^2 = QC \cdot QD$. Substituting in our known values, we obtain $QA(QA+10) = 12^2 = QC(QC+7)$. Solving these quadratics, we get that $QA = 8$ and $QC = 9$.

We can see that $\frac{AQ}{DQ} = \frac{CQ}{BQ}$ and that $\angle AQD = \angle CQB$, so $QAD \sim QCB$. (Alternatively, going back to the equality $QA \cdot QB = QC \cdot QD$, we realize that this is just a Power of a Point theorem on the quadrilateral $ABDC$, and so this quadrilateral is cyclic. This implies that $\angle ADQ = \angle ADC = \angle ABC = \angle QBC$.) Thus, $\frac{AD}{BC} = \frac{AQ}{QC} = \frac{8}{9}$.



10. [**8**] Let $r_1, r_2, \ldots, r_7$ be the distinct complex roots of the polynomial $P(x) = x^7 - 7$. Let

$$K = \prod_{1 \le i < j \le 7} (r_i + r_j),$$

that is, the product of all numbers of the form $r_i + r_j$, where i and j are integers for which $1 \le i < j \le 7$. Determine the value of K^2.

Answer: $\boxed{117649}$ We first note that $x^7 - 7 = (x-r_1)(x-r_2)\cdots(x-r_7)$, which implies, replacing x by $-x$ and taking the negative of the equation, that $(x+r_1)(x+r_2)\cdots(x+r_7) = x^7 + 7$. Also note that the product of the r_i is just the constant term, so $r_1 r_2 \cdots r_7 = 7$.

Now, we have that

$$\begin{aligned}
2^7 \cdot 7 \cdot K^2 &= (\prod_{i=1}^{7} 2r_i)K^2 \\
&= \prod_{i=1}^{7} 2r_i \prod_{1\le i<j\le 7} (r_i + r_j)^2 \\
&= \prod_{1\le i=j\le 7} (r_i + r_j) \prod_{1\le i<j\le 7} (r_i + r_j) \prod_{1\le j<i\le 7} (r_i + r_j) \\
&= \prod_{1\le i,j\le 7} (r_i + r_j) \\
&= \prod_{i=1}^{7} \prod_{j=1}^{7} (r_i + r_j).
\end{aligned}$$

However, note that for any fixed i, $\prod_{j=1}^{7} (r_i + r_j)$ is just the result of substuting $x = r_i$ into $(x + r_1)(x + r_2) \cdots (x + r_7)$. Hence,

$$\prod_{j=1}^{7} (r_i + r_j) = r_i^7 + 7 = (r_i^7 - 7) + 14 = 14.$$

Therefore, taking the product over all i gives 14^7, which yields $K^2 = 7^6 = 117649$.